

η - η' mixing in $U(3)$ chiral perturbation theory

N. Beisert^a and B. Borasoy^b

Physik Department, Technische Universität München, D-85747 Garching, Germany

Received: 12 March 2001 / Revised version: 19 June 2001

Communicated by A. Schäfer

Abstract. We investigate η - η' mixing in infrared regularized $U(3)$ chiral perturbation theory by calculating the η and η' masses up to one-loop order. From this analysis it becomes obvious that even at leading order η - η' -mixing does not obey the usually assumed one-mixing-angle scheme if large N_c counting rules are not employed.

PACS. 12.39.Fe Chiral Lagrangians – 14.40.-n Mesons

1 Introduction

The η - η' mixing has been the subject of many investigations, see, *e.g.*, [1–14]. Both particles can be described as mixtures of the octet component η_8 and its singlet counterpart η_0 . The η_8 , which is a member of the octet of the pseudoscalar mesons (π, K, η_8), differs from the singlet η_0 in a substantial way: it is a Goldstone boson whose mass vanishes in the limit of zero quark masses, while the η_0 is not due to the axial $U(1)$ anomaly.

Phenomenologically, however, the situation for the η - η' mixing still remains to be settled. Most of the investigations on this subject introduce one single mixing angle and extract a value from different kinds of data. These are, *e.g.*, the anomalous $\eta, \eta' \rightarrow \gamma\gamma$ [1, 2], decays of J/Ψ [3–5], electromagnetic decays of vector and pseudoscalar mesons [6], only to name a few. The values obtained in these investigations range from -13° [6] to -22° [2]. On the other hand, the Gell-Mann–Okubo mass formula for the pseudoscalar mesons yields a mixing angle of -10° [7].

More recently, a two-mixing-angle scheme has been proposed by Kaiser and Leutwyler [8–10] for the calculation of the pseudoscalar decay constants in large N_c chiral perturbation theory. The two-angle scenario has been adopted in a phenomenological analysis on the two-photon decay widths of the η and η' , the $\eta\gamma$ and $\eta'\gamma$ transition form factors, radiative J/Ψ decays, as well as on the decay constants of the pseudoscalar mesons [11, 12]. The authors observe that within their phenomenological approach the assumption of one mixing angle is not in agreement with experiment, whereas the two-mixing-angle scheme leads to a very good description of the data. These

two different mixing angles have been interpreted as one energy-dependent η - η' mixing angle in [13], where electromagnetic couplings between lowest-lying vector and pseudoscalar mesons were studied. As pointed out in these investigations, the analysis with two different mixing angles leads to a more coherent picture than the canonical treatment with a single angle. In particular, the calculation of the pseudoscalar decay constants within the framework of large N_c chiral perturbation theory *requires* two different mixing angles [8]. (A similar investigation was performed in [14] but with a different parametrization.)

Recently, it has been shown in [15] that the η' can be included in a systematic way in chiral perturbation theory without employing $1/N_c$ counting rules. The loop integrals are evaluated using infrared regularization, which preserves Lorentz and chiral symmetry [16]. However, in [15] it was assumed that the η - η' mixing follows at lowest order in symmetry-breaking the one-mixing-angle pattern, *i.e.* the mixing is described only by one mixing angle and its value was assumed to be -20° .

The purpose of this work is to critically investigate η - η' mixing up to one-loop order in infrared regularized $U(3)$ chiral perturbation theory which provides a systematic counting scheme. Within this approach, loops start contributing at next-to-leading order while they are a next-to-next-to-leading order effect in large N_c chiral perturbation theory.

We start in the next section by presenting the effective Lagrangian and η - η' mixing at lowest order. The next-to-leading order calculation within this counting scheme including one-loop diagrams is presented in sect. 3. We also compare this approach with a scheme that takes only loops with Goldstone bosons into account omitting any propagation of an η' inside the loop. Section 4 contains our results and we conclude with a summary in sect. 5.

^a e-mail: nbeisert@physik.tu-muenchen.de

^b e-mail: borasoy@physik.tu-muenchen.de

2 η - η' mixing at leading order

In this section, we present η - η' mixing at lowest order in the framework of infrared regularized $U(3)$ chiral perturbation theory. Note that we do not make use of $1/N_c$ counting rules. The effective Lagrangian for the pseudoscalar meson nonet (π, K, η_8, η_0) reads, up to second order in the derivative expansion [9, 10, 15]¹,

$$\begin{aligned} \mathcal{L}^{(0+2)} = & -V_0 + V_1 \langle D_\mu U^\dagger D^\mu U \rangle + V_2 \langle U^\dagger \chi + \chi^\dagger U \rangle \\ & + iV_3 \langle U^\dagger \chi - \chi^\dagger U \rangle + V_4 \langle U^\dagger D^\mu U \rangle \langle U^\dagger D_\mu U \rangle \\ & + iV_5 D^\mu \theta \langle U^\dagger D_\mu U \rangle + V_6 D^\mu \theta D_\mu \theta, \end{aligned} \quad (1)$$

where U is a unitary 3×3 matrix containing the Goldstone boson octet (π, K, η_8) and the η' . Its dependence on η_8 and η_0 is given by

$$U = \exp(\text{diag}(1, 1, -2) \cdot i\eta_8/\sqrt{3}f + i\sqrt{2}\eta_0/\sqrt{3}f + \dots). \quad (2)$$

The expression $\langle \dots \rangle$ denotes the trace in flavor space, f is the pion decay constant in the chiral limit and the quark mass matrix $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ enters in the combination $\chi = 2B\mathcal{M}$, with $B = -\langle 0|\bar{q}q|0\rangle/f^2$ being the order parameter of the spontaneous symmetry violation. The external field θ is the QCD vacuum angle, which will be set to zero throughout this discussion. The covariant derivatives are defined by

$$\begin{aligned} D_\mu U &= \partial_\mu U - i(v_\mu + \tilde{a}_\mu)U + iU(v_\mu - \tilde{a}_\mu), \\ D_\mu \theta &= \frac{\sqrt{6\lambda}}{f} \partial_\mu \theta + 2\langle \tilde{a}_\mu \rangle. \end{aligned} \quad (3)$$

They are defined in such a way that all the dependence on the running scale of QCD due to the anomalous dimension of the singlet axial current $A_\mu^0 = \frac{1}{2}\bar{q}\gamma_\mu\gamma_5q$ is absorbed into the prefactor $\sqrt{\lambda}$, cf. [15] for details. Due to its scale dependence, $\sqrt{\lambda}$ cannot be determined from experiment, and all quantities involving it are unphysical. The axial-vector connection \tilde{a}_μ is defined as

$$\tilde{a}_\mu = a_\mu + \frac{\sqrt{6\lambda} - f}{3f} \langle a_\mu \rangle. \quad (4)$$

which is the scale-independent combination of the octet and singlet parts of the external axial-vector field a_μ .

For $\theta = 0$ the coefficients V_i are functions of η_0 , $V_i(\eta_0/f)$, and can be expanded in terms of this variable. At a given order of derivatives of the meson fields U and insertions of the quark mass matrix \mathcal{M} , one obtains an infinite string of increasing powers of η_0 with couplings which are not fixed by chiral symmetry. Parity conservation implies that the V_i are all even functions of η_0 except V_3 , which is odd, and $V_1(0) = V_2(0) = V_1(0) - 3V_4(0) = \frac{1}{4}f^2$ gives the correct normalization for the quadratic terms of

the mesons. The potentials V_i are expanded in the singlet field η_0

$$\begin{aligned} V_i \left[\frac{\eta_0}{f} \right] &= v_i^{(0)} + v_i^{(2)} \frac{\eta_0^2}{f^2} + v_i^{(4)} \frac{\eta_0^4}{f^4} + \dots \\ &\text{for } i = 0, 1, 2, 4, 5, 6, \\ V_3 \left[\frac{\eta_0}{f} \right] &= v_3^{(1)} \frac{\eta_0}{f} + v_3^{(3)} \frac{\eta_0^3}{f^3} + \dots, \end{aligned} \quad (5)$$

with expansion coefficients $v_i^{(j)}$ to be determined phenomenologically. In the present investigation we work in the isospin limit $m_u = m_d = \hat{m}$ and, therefore, only η - η' mixing occurs. One observes terms quadratic in the meson fields that contain the factor $\eta_0\eta_8$. Such terms arise from the explicitly chiral symmetry-breaking operators $V_2 \langle U^\dagger \chi + \chi^\dagger U \rangle + iV_3 \langle U^\dagger \chi - \chi^\dagger U \rangle$ and read

$$-\frac{8\sqrt{2}}{3f^2} \left(\frac{1}{4}f^2 - \frac{1}{2}\sqrt{6}v_3^{(1)} \right) B(\hat{m} - m_s)\eta_0\eta_8. \quad (6)$$

However, these are not the only η_0 - η_8 mixing terms arising at second chiral order. Terms from $\mathcal{L}^{(4)}$, the Lagrangian at fourth chiral order, which is presented in Appendix A, will also contribute to the mass matrix at second chiral order. This can be seen as follows. Consider the terms in $\mathcal{L}^{(4)}$ with one or two derivatives of the singlet field and an insertion of the quark mass matrix χ . They are given by [15, 17]

$$\begin{aligned} \mathcal{L}^{(4)} = & \dots + \frac{2}{3f^2} (3\beta_4 + \beta_5 - 9\beta_{17} + 3\beta_{18}) \\ & \cdot e^{-i\sqrt{6}\eta_0/(3f)} D_\mu \eta_0 D^\mu \eta_0 \langle \hat{U}^\dagger \chi \rangle + \text{h.c.} \\ & + \frac{i\sqrt{6}}{3f} (2\beta_5 + 3\beta_{18}) e^{-i\sqrt{6}\eta_0/(3f)} D_\mu \eta_0 \langle D^\mu \hat{U}^\dagger \chi \rangle \\ & + \text{h.c.} + \dots, \end{aligned} \quad (7)$$

where we have kept the notation from [15] and $\hat{U} = (\det U)^{-1/3}U$ contains only Goldstone boson fields. The β_i are functions of the singlet field η_0 and can be expanded as in eq. (5). They contribute to the part of the effective Lagrangian quadratic in the η_8 and η_0 which has the generic form

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \partial_\mu \eta_i (\delta_{ij} + K_{ij}^{(2)}) \partial^\mu \eta_j - \frac{1}{2} \eta_i (M_{ij}^{(0)} + M_{ij}^{(2)}) \eta_j, \quad (8) \\ & i, j = 0, 8, \end{aligned}$$

where the superscripts for the matrices K and M denote the chiral power. (We restrict ourselves to the η - η' system since pions and kaons decouple in the isospin limit.) Choosing K and M in a symmetric form, one obtains from

¹ If one prefers, one can transform the V_5 term away. Here we keep it for completeness.

eqs. (1) and (7) the non-vanishing coefficients

$$\begin{aligned}
 M_{00}^{(0)} &= \overset{\circ}{m}_0^2, \\
 M_{88}^{(2)} &= \overset{\circ}{m}_8^2 + \frac{1}{2}\overset{\circ}{m}_\Delta^2, \\
 M_{08}^{(2)} &= -2\sqrt{2}\tilde{v}_2^{(1)}\overset{\circ}{m}_8^2/f^2, \\
 M_{00}^{(2)} &= 4\tilde{v}_2^{(2)}\overset{\circ}{m}_8^2/f^2, \\
 K_{08}^{(2)} &= -4\sqrt{2}\beta_{5,18}\overset{\circ}{m}_\Delta^2/f^2, \\
 K_{00}^{(2)} &= 8\beta_{4,5,17,18}\overset{\circ}{m}_8^2/f^2.
 \end{aligned} \tag{9}$$

Here we have made the following abbreviations for combinations of constants that repeatedly occur:

$$\begin{aligned}
 \overset{\circ}{m}_0^2 &= \frac{2v_0^{(2)}}{f^2}, \\
 \overset{\circ}{m}_8^2 &= \frac{2}{3}B(2\hat{m} + m_s), \\
 \overset{\circ}{m}_\Delta^2 &= \frac{4}{3}B(m_s - \hat{m}), \\
 \tilde{v}_2^{(1)} &= \frac{1}{4}f^2 - \frac{1}{2}\sqrt{6}v_3^{(1)}, \\
 \tilde{v}_2^{(2)} &= \frac{1}{4}f^2 - \sqrt{6}v_3^{(1)} - 3v_2^{(2)}, \\
 \beta_{5,18} &= \beta_5^{(0)} + \frac{3}{2}\beta_{18}^{(0)}, \\
 \beta_{4,5,17,18} &= 3\beta_4^{(0)} + \beta_5^{(0)} - 9\beta_{17}^{(0)} + 3\beta_{18}^{(0)}.
 \end{aligned} \tag{10}$$

The mass of the η' in the chiral limit is denoted by $\overset{\circ}{m}_0$, $\overset{\circ}{m}_8^2$ is the mean mass squared of the octet, $\overset{\circ}{m}_\Delta^2$ describes the mass splitting of the octet. Both combinations $\tilde{v}_2^{(1)}$ and $\tilde{v}_2^{(2)}$ approach $\frac{1}{4}f^2$ in the large N_c limit. The scale dependence of the renormalized β_i parameters cancels in the combinations $\beta_{5,18}$ and $\beta_{4,5,17,18}$.

The wave functions must be renormalized in order to acquire the canonical form for the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\partial_\mu\eta'\partial^\mu\eta' - \frac{1}{2}\overset{\circ}{m}_\eta^2\eta^2 - \frac{1}{2}\overset{\circ}{m}_{\eta'}^2\eta'^2. \tag{11}$$

To second order this is achieved by the transformation $(\eta_8, \eta_0)^T = (1 + R_0^{(2)})(\eta, \eta')^T$, with

$$1 + R_0^{(2)} = \begin{pmatrix} 1 & M_{08}^{(2)}/M_{00}^{(0)} - K_{08}^{(2)} \\ -M_{08}^{(2)}/M_{00}^{(0)} & 1 - \frac{1}{2}K_{00}^{(2)} \end{pmatrix}. \tag{12}$$

The off-diagonal elements of this transformation describe mixing between the fields η and η' . Even in leading order the two off-diagonal elements are different in contradistinction to large N_c chiral perturbation theory (cf. [9, 17]), where the sum of both off-diagonal elements vanishes in leading order. There the term $K_{08}^{(2)}$ is of higher order than $M_{08}^{(2)}/M_{00}^{(0)}$ and it is justified to use just one mixing angle. In our approach we have two different off-diagonal elements which leads directly to two different mixing angles.

The two off-diagonal elements are not the two mixing angles occurring in the pseudoscalar decay constants of the

η and the η' , although they are closely related. In leading order the η_i fields couple to \tilde{a}_i^μ with strength f for $i = 8$ and $f_0 = \sqrt{6}\lambda(1 + 6v_5^{(0)}/f^2)$ for $i = 0$, see eq. (26). After the transformation to η, η' fields the coupling matrix can be written as $\text{diag}(f, f_0)(1 + R_0^{(2)})$. We will see later that there are also loop corrections to the coupling matrix in second chiral order, however, $R_0^{(2)}$ involves two amplitudes and two mixing angles already at tree level. For the full results, see eqs. (28) and (29).

Note also that $(1 + R_0^{(2)})$ in eq. (12) is not the complete wave function renormalization to second chiral order; there are corrections from loops and LECs which are presented in the next section. However, they do not affect the masses at this order and could be dropped so far. The full matrix is given in eq. (22).

After the transformation the masses can be read off from the Lagrangian, they are given by $\overset{\circ}{m}_\eta^2 = M_{88}^{(2)}$ and $\overset{\circ}{m}_{\eta'}^2 = M_{00}^{(0)} + M_{00}^{(2)} - M_{00}^{(0)}K_{00}^{(2)}$. Expressed in $U(3)$ parameters the masses at second chiral order are

$$\begin{aligned}
 \overset{\circ}{m}_\pi^2 &= \overset{\circ}{m}_8^2 - \frac{1}{2}\overset{\circ}{m}_\Delta^2, \\
 \overset{\circ}{m}_K^2 &= \overset{\circ}{m}_8^2 + \frac{1}{4}\overset{\circ}{m}_\Delta^2, \\
 \overset{\circ}{m}_\eta^2 &= \overset{\circ}{m}_8^2 + \frac{1}{2}\overset{\circ}{m}_\Delta^2, \\
 \overset{\circ}{m}_{\eta'}^2 &= \overset{\circ}{m}_0^2 + \frac{4}{f^2}\overset{\circ}{m}_8^2(\tilde{v}_2^{(2)} - 2\overset{\circ}{m}_0^2\beta_{4,5,17,18}),
 \end{aligned} \tag{13}$$

where we have included the π and K masses for completeness. Note that the Gell-Mann–Okubo mass relation is satisfied in leading order.

3 Inclusion of loops

We proceed by investigating η - η' mixing in the calculation of the η and η' masses at next-to-leading order. To this order contributions both from one-loop graphs and higher-order contact terms must be taken into account. The fourth-order Lagrangian is given by

$$\mathcal{L}^{(4)} = \sum_k \beta_k O_k, \tag{14}$$

where the fourth-order operators are given in Appendix A. In the present work the contributing operators O_k are those with $k = 0, \dots, 8, 12, 13, 14, 15, 25, 26$. We have decided to include the β_0 term, although there is a Cayley-Hamilton matrix identity that enables one to remove the term leading to modified coefficients β_i , $i = 1, 2, 3, 13, 14, 15, 16$ [17]. (It is actually more convenient to eliminate one of the OZI-violating terms β_{14} , β_{15} or β_{16} , see [18].) Here we do not make use of the Cayley-Hamilton identity and keep all couplings in order to present the most general expressions in terms of these parameters. One can then drop one of the β_i involved in the Cayley-Hamilton identity at any stage of the calculation. Furthermore, one-loop diagrams from the Lagrangian in eq. (1) contribute at this order. It is crucial to employ infrared regularization

in the evaluation of the loop diagrams if one does not implement large N_c counting rules. Otherwise, the inclusion of η' loops would spoil the counting scheme and in general higher loops with an arbitrary number of η' -propagators will contribute to lower chiral orders. This is similar to the situation in the relativistic framework of dimensionally regularized baryon chiral perturbation theory. Using infrared regularization allows for a chiral counting scheme while preserving chiral invariance [16]. The loop diagrams are usually divergent and must be renormalized by counterterms of arbitrarily high order. This cannot be done in practice and one neglects these counterterm polynomials [16]. We will proceed in a similar way, restricting ourselves to the calculation of the chiral logarithms and checking the scale dependence of the non-analytic portions of the chiral loops by varying the scale. We will assume that the divergences have been absorbed by a redefinition of the LECs and use the same notation for the renormalized coupling constants. In the present calculation this amounts to keeping only the chiral logarithms of the loops with the Goldstone bosons. (A more rigorous investigation of renormalization is provided within a modified framework in the subsequent section. The advantage of this approach is that the complete renormalization of the one-loop function can be performed.)

The effective Lagrangian at one-loop order quadratic in the fields η and η' has the form

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \partial_\mu \eta \left[1 + T_{88}^{(2)} \right] \partial^\mu \eta \\ & - \frac{1}{2} \eta \left[M_{88}^{(2)} - (M_{08}^{(2)})^2 / M_{00}^{(0)} + M_{88}^{(4)} \right] \eta \\ & + \frac{1}{2} \partial_\mu \eta' \left[1 + T_{00}^{(4)} - \frac{3}{4} (K_{00}^{(2)})^2 - (K_{08}^{(2)})^2 \right. \\ & \quad \left. + (M_{08}^{(2)} / M_{00}^{(0)})^2 \right] \partial^\mu \eta' \\ & - \frac{1}{2} \eta' \left[M_{00}^{(0)} + M_{00}^{(2)} + M_{00}^{(4)} + 2(M_{08}^{(2)})^2 / M_{00}^{(0)} \right. \\ & \quad \left. - K_{00}^{(2)} (M_{00}^{(0)} + M_{00}^{(2)}) - 2K_{08}^{(2)} M_{08}^{(2)} \right. \\ & \quad \left. + \frac{1}{4} (K_{00}^{(2)})^2 M_{00}^{(0)} \right] \eta'. \end{aligned} \quad (15)$$

We have not shown the off-diagonal elements proportional to $\eta\eta'$ since these do not contribute to the masses at fourth chiral order. The term $T_{88}^{(2)}$ is the fourth-order correction arising from one-loop diagrams with a $V_1(0)$ vertex and contact terms from $\mathcal{L}^{(4)}$ in eq. (14)

$$T_{88}^{(2)} = \frac{1}{f^2} (24\beta_4^{(0)} \overset{\circ}{m}_8^2 + 8\beta_5^{(0)} \overset{\circ}{m}_\eta^2 - \Delta_K), \quad (16)$$

with $\Delta_\phi = \overset{\circ}{m}_\phi^2 / (16\pi^2) \ln(\overset{\circ}{m}_\phi^2 / \mu^2)$ and μ the scale introduced in infrared regularization. In order to account for all contributions to the masses at fourth chiral order, $T_{00}^{(4)}$ must include two-loop diagrams with vertices from $\mathcal{L}^{(2)}$, one-loop graphs from $\mathcal{L}^{(4)}$ and contact terms from $\mathcal{L}^{(6)}$. Possible two-loop diagrams are the sunset diagram and double tadpoles. It turns out that they do not contribute

to the order we are working if infrared regularization is employed. The only contributions to $T_{00}^{(4)}$ arise from contact terms of $\mathcal{L}^{(6)}$ and from one-loop diagrams — tadpoles in our case — with $\mathcal{L}^{(4)}$ vertices. An enumeration of all possible counterterms in $\mathcal{L}^{(6)}$ is beyond the scope of the present investigation. We will only need terms proportional to $D^\mu \eta_0 D_\mu \eta_0$, multiplied by chirally invariant combinations of two quark mass matrices. Setting $U = 1$ the only two independent combinations are $\langle \chi \rangle^2$ and $\langle \chi^2 \rangle$, and we summarize all contributing terms to $T_{00}^{(4)}$ in the following Lagrangian:

$$\begin{aligned} \mathcal{L}^{(6)} = & \dots + \frac{1}{2f^2} \left[\gamma_1 (\overset{\circ}{m}_8^2)^2 + \gamma_2 (2(\overset{\circ}{m}_8^2)^2 + (\overset{\circ}{m}_\Delta^2)^2) \right] \\ & \cdot D_\mu \eta_0 D^\mu \eta_0 + \dots \end{aligned} \quad (17)$$

Including these terms the results for $T_{00}^{(4)}$, $M_{88}^{(4)}$ and $M_{00}^{(4)}$ read

$$\begin{aligned} T_{00}^{(4)} = & \frac{4}{f^4} (2\beta_0^{(0)} + 4\beta_1^{(0)} + \beta_2^{(0)} + 2\beta_3^{(0)} - \frac{2}{3}\beta_{4,5,17,18} \\ & - 3\beta_{13}^{(0)} - 6\beta_{14}^{(0)} - \frac{3}{2}\beta_{15}^{(0)}) \\ & \cdot (3\overset{\circ}{m}_\pi^2 \Delta_\pi + 4\overset{\circ}{m}_K^2 \Delta_K + \overset{\circ}{m}_\eta^2 \Delta_\eta) \\ & + \frac{1}{f^2} (\gamma_1 (\overset{\circ}{m}_8^2)^2 + \gamma_2 (\overset{\circ}{m}_\Delta^2)^2), \\ M_{88}^{(4)} = & \frac{1}{f^2} (-\frac{1}{2}\overset{\circ}{m}_\pi^2 \Delta_\pi + \frac{1}{3}\overset{\circ}{m}_\pi^2 \Delta_K - \frac{8}{9}\overset{\circ}{m}_K^2 \Delta_\eta + \frac{7}{18}\overset{\circ}{m}_\pi^2 \Delta_\eta) \\ & + \frac{8}{f^2} (6\beta_6^{(0)} \overset{\circ}{m}_\eta^2 \overset{\circ}{m}_8^2 + 3\beta_7^{(0)} (\overset{\circ}{m}_\Delta^2)^2 \\ & + 2\beta_8^{(0)} (\overset{\circ}{m}_\eta^2)^2 + \beta_8^{(0)} (\overset{\circ}{m}_\Delta^2)^2), \\ M_{00}^{(4)} = & \frac{4}{3f^4} (-\tilde{v}_2^{(2)} - 3v_1^{(2)}) (3\overset{\circ}{m}_\pi^2 \Delta_\pi + 4\overset{\circ}{m}_K^2 \Delta_K + \overset{\circ}{m}_\eta^2 \Delta_\eta) \\ & + \frac{24}{f^2} (2\beta_6^{(0)} - \sqrt{6}\beta_{26}^{(1)} - 3\beta_6^{(2)} + 2\beta_7^{(0)}) (\overset{\circ}{m}_8^2)^2 \\ & + \frac{1}{f^2} (8\beta_8^{(0)} - 4\sqrt{6}\beta_{25}^{(1)} - 6\beta_8^{(2)} - 3\beta_{12}^{(2)}) \\ & \cdot (2(\overset{\circ}{m}_8^2)^2 + (\overset{\circ}{m}_\Delta^2)^2). \end{aligned} \quad (18)$$

The other terms in eq. (15) are higher-order corrections from the transformation in eq. (12) which were not presented for the calculation at second chiral order.

The transformations

$$\begin{aligned} \eta & \rightarrow \left[1 - \frac{1}{2} T_{88}^{(2)} \right] \eta, \\ \eta' & \rightarrow \left[1 - \frac{1}{2} T_{00}^{(4)} + \frac{3}{8} (K_{00}^{(2)})^2 + (K_{08}^{(2)})^2 \right. \\ & \quad \left. - \frac{1}{2} (M_{08}^{(2)} / M_{00}^{(0)})^2 \right] \eta', \end{aligned} \quad (19)$$

bring the diagonal elements of the kinetic terms into the canonical form and change the masses

$$\begin{aligned} m_\eta^2 &= M_{88}^{(2)} + M_{88}^{(4)} - T_{88}^{(2)} M_{88}^{(2)} - (M_{08}^{(2)})^2 / M_{00}^{(2)}, \\ m_{\eta'}^2 &= M_{00}^{(0)} + M_{00}^{(2)} + M_{00}^{(4)} - T_{00}^{(4)} M_{00}^{(0)} \\ &\quad - K_{00}^{(2)} (M_{00}^{(0)} + M_{00}^{(2)} - M_{00}^{(0)} K_{00}^{(2)}) \\ &\quad + (M_{08}^{(2)} - M_{00}^{(0)} K_{08}^{(2)})^2 / M_{00}^{(0)}. \end{aligned} \quad (20)$$

Substitution of these terms gives

$$\begin{aligned} m_\pi^2 &= \overset{\circ}{m}_\pi^2 \left[1 + 8(2\beta_8^{(0)} - \beta_5^{(0)}) \frac{\overset{\circ}{m}_\pi^2}{f^2} \right. \\ &\quad \left. + 24(2\beta_6^{(0)} - \beta_4^{(0)}) \frac{\overset{\circ}{m}_8^2}{f^2} + \frac{\frac{1}{2}\Delta_\pi - \frac{1}{6}\Delta_\eta}{f^2} \right], \\ m_K^2 &= \overset{\circ}{m}_K^2 \left[1 + 8(2\beta_8^{(0)} - \beta_5^{(0)}) \frac{\overset{\circ}{m}_K^2}{f^2} \right. \\ &\quad \left. + 24(2\beta_6^{(0)} - \beta_4^{(0)}) \frac{\overset{\circ}{m}_8^2}{f^2} + \frac{\frac{1}{3}\Delta_\eta}{f^2} \right], \\ m_\eta^2 &= \overset{\circ}{m}_\eta^2 \left[1 + 8(2\beta_8^{(0)} - \beta_5^{(0)}) \frac{\overset{\circ}{m}_\eta^2}{f^2} + 24(2\beta_6^{(0)} - \beta_4^{(0)}) \frac{\overset{\circ}{m}_8^2}{f^2} \right] \\ &\quad + \frac{(\overset{\circ}{m}_\Delta^2)}{f^2} \left[8\beta_8^{(0)} + 24\beta_7^{(0)} - \frac{8(\tilde{v}_2^{(1)})^2}{f^2 \overset{\circ}{m}_0^2} \right] \\ &\quad + \frac{-\frac{1}{2}\overset{\circ}{m}_\pi^2 \Delta_\pi + \frac{4}{3}\overset{\circ}{m}_K^2 \Delta_K + \frac{7}{18}\overset{\circ}{m}_\pi^2 \Delta_\eta - \frac{8}{9}\overset{\circ}{m}_K^2 \Delta_\eta}{f^2}, \\ m_{\eta'}^2 &= \overset{\circ}{m}_{\eta'}^2 + \frac{8\overset{\circ}{m}_8^2 (\overset{\circ}{m}_0^2 - \overset{\circ}{m}_{\eta'}^2) \beta_{4,5,17,18}}{f^2} \\ &\quad + \frac{8(\overset{\circ}{m}_\Delta^2)^2 (\tilde{v}_2^{(1)} - 2\overset{\circ}{m}_0^2 \beta_{5,18})^2}{f^4 \overset{\circ}{m}_0^2} \\ &\quad + \frac{3\overset{\circ}{m}_\pi^2 \Delta_\pi + 4\overset{\circ}{m}_K^2 \Delta_K + \overset{\circ}{m}_\eta^2 \Delta_\eta}{f^4} \left[-\frac{4}{3}\tilde{v}_2^{(2)} - 4v_1^{(2)} \right. \\ &\quad \left. + \frac{8}{3}\overset{\circ}{m}_0^2 \beta_{4,5,17,18} + \overset{\circ}{m}_0^2 (-8\beta_0^{(0)} - 16\beta_1^{(0)} \right. \\ &\quad \left. - 4\beta_2^{(0)} - 8\beta_3^{(0)} + 12\beta_{13}^{(0)} + 24\beta_{14}^{(0)} + 6\beta_{15}^{(0)}) \right] \\ &\quad + (48\beta_6^{(0)} - 24\sqrt{6}\beta_{26}^{(1)} - 72\beta_6^{(2)} + 48\beta_7^{(0)} - \overset{\circ}{m}_0^2 \gamma_1) \\ &\quad \cdot \frac{(\overset{\circ}{m}_8^2)^2}{f^2} + \frac{2(\overset{\circ}{m}_8^2)^2 + (\overset{\circ}{m}_\Delta^2)^2}{f^2} \\ &\quad \cdot (8\beta_8^{(0)} - 4\sqrt{6}\beta_{25}^{(1)} - 6\beta_8^{(2)} - 3\beta_{12}^{(2)} - \overset{\circ}{m}_0^2 \gamma_2). \end{aligned} \quad (21)$$

This completes the calculation of the η and η' masses up to fourth chiral order. In [15] it was assumed that the η - η' mixing follows the one-mixing-angle scheme and some of the terms in eq. (20) have been neglected. (This was sufficient in order to establish infrared regularized $U(3)$ chiral

perturbation theory, and the main purpose of this paper was to show that the chiral series for the masses and decay constants converge faster than in the dimensionally regularized theory.) A rigorous treatment of the masses up to fourth chiral order, however, requires the transformation $(\eta_8, \eta_0)^T = (1 + R^{(2)} + R^{(4)})(\eta, \eta')^T$, with

$$1 + R^{(2)} = \begin{pmatrix} 1 - \frac{1}{2}T_{88}^{(2)} & M_{08}^{(2)}/M_{00}^{(0)} - K_{08}^{(2)} \\ -M_{08}^{(2)}/M_{00}^{(0)} & 1 - \frac{1}{2}K_{00}^{(2)} \end{pmatrix}, \quad (22)$$

where we have presented only the terms up to second chiral order for brevity. (The fourth-order terms only give contributions to $m_{\eta'}$ and scattering processes involving several η' .) This generalizes eq. (12), and the entries of $R^{(2)}$ are given by

$$\begin{aligned} R_\pi^{(2)} &= (-12\overset{\circ}{m}_8^2 \beta_4^{(0)} - 4\overset{\circ}{m}_\pi^2 \beta_5^{(0)} + \frac{1}{3}\Delta_\pi + \frac{1}{6}\Delta_K) / f^2, \\ R_K^{(2)} &= (-12\overset{\circ}{m}_8^2 \beta_4^{(0)} - 4\overset{\circ}{m}_K^2 \beta_5^{(0)} + \frac{1}{8}\Delta_\pi + \frac{1}{4}\Delta_K + \frac{1}{8}\Delta_\eta) / f^2, \\ R_{8\eta}^{(2)} &= (-12\overset{\circ}{m}_8^2 \beta_4^{(0)} - 4\overset{\circ}{m}_\eta^2 \beta_5^{(0)} + \frac{1}{2}\Delta_K) / f^2, \\ R_{8\eta'}^{(2)} &= 2\sqrt{2}\overset{\circ}{m}_\Delta^2 (2\overset{\circ}{m}_0^2 \beta_{5,18} - \tilde{v}_2^{(1)}) / f^2 \overset{\circ}{m}_0^2, \\ R_{0\eta}^{(2)} &= 2\sqrt{2}\overset{\circ}{m}_\Delta^2 \tilde{v}_2^{(1)} / f^2 \overset{\circ}{m}_0^2, \\ R_{0\eta'}^{(2)} &= -4\overset{\circ}{m}_8^2 \beta_{4,5,17,18} / f^2, \end{aligned} \quad (23)$$

where the expressions for the pions and kaons have been included for completeness. The matrix $R^{(2)}$ constitutes one of our main results since it will be needed in all one-loop calculations.

3.1 Renormalization

From the above formulas it becomes apparent that η' loops do not contribute at this order in infrared regularization. The tadpole which is the only one-loop graph in the present investigation vanishes in the case of the η' . A similar observation is made in [19] where both tadpoles and chiral unitarity corrections have been evaluated for the hadronic decay $\eta' \rightarrow \eta\pi\pi$. Employing infrared regularization, loops with an η' contribute at higher orders than pure Goldstone boson loops. For the processes considered so far the infrared physics stemming from the propagation of an η' inside the loop is suppressed by one chiral order and therefore beyond the working accuracy of [19] and the present investigation. At this order it is therefore equivalent to a scheme in which the η' is not taken into account at all in loops but rather treated first as a background field. Only Goldstone boson loops occur within this approach and they are calculated employing dimensional regularization. After the evaluation of the loops the η' field can be dealt with as a propagating field. The main advantage of such a framework is given by the complete renormalization of the one-loop functional which cannot be undertaken in infrared regularization since it involves the renormalization of counterterms of infinite order. In addition to being an alternative approach for describing

η' physics at low energies it provides a check on the renormalization of the Goldstone boson integrals in infrared regularization.

In the appendices, we present a list of all operators of the fourth-order Lagrangian and the complete renormalization of the one-loop functional of the Goldstone boson loops. We would like to point out that our results for the renormalization differ substantially from those in [14] since within this work the authors treated the η' on the same footing as the Goldstone bosons and included the η' inside loops.

4 Results

The decay constant f is taken to be 88 MeV, the value of the pion decay constant in the chiral limit [20]. The quark mass matrix is chosen to fit $\overset{\circ}{m}_\pi = 138$ MeV and $\overset{\circ}{m}_K = 496$ MeV. We take the values of $\beta_i^{(0)} = \beta_i^{SU(3)}$ from ordinary $SU(3)$ chiral perturbation theory [21] unless stated otherwise.

4.1 Masses

First we investigate the mass of the η' at second chiral order in eq. (13)

$$\overset{\circ}{m}_{\eta'}^2 = \overset{\circ}{m}_0^2 + \frac{4}{f^2} \overset{\circ}{m}_8^2 (\tilde{v}_2^{(2)} - 2\overset{\circ}{m}_0^2 \beta_{4,5,17,18}). \quad (24)$$

The phenomenological values for $\beta_{17}^{(0)}$ and $\beta_{18}^{(0)}$ are not known, but they are OZI-violating corrections to $\beta_5^{(0)}$. Assuming that they are suppressed at the scale $\mu = m_\rho$, *i.e.* $|\beta_{17}^{(0)}(m_\rho)|, |\beta_{18}^{(0)}(m_\rho)| \ll |\beta_5^{(0)}(m_\rho)|$, we can roughly estimate $\beta_{4,5,17,18} \approx 0.5 \times 10^{-3}$. However, this result is very sensitive to the scale at which the OZI rule has been applied: at $\mu = m_\eta$ it yields $\beta_{4,5,17,18} \approx 2.0 \times 10^{-3}$.

In order to obtain a bound for $\beta_{4,5,17,18}$, we will consider the dependence of $\overset{\circ}{m}_{\eta'}^2$ on $\overset{\circ}{m}_0^2$. We assume that the proportionality factor $(1 - 8\beta_{4,5,17,18} \overset{\circ}{m}_8^2 / f^2)$ converges reasonably fast, *i.e.*, that the second term is at most a 25% correction to the leading order. This gives the limit $|\beta_{4,5,17,18}| < 1.5 \times 10^{-3}$.

Next we assume $0 < \tilde{v}_2^{(2)} < \frac{1}{2}f^2$ in agreement with large N_c considerations and solve $\overset{\circ}{m}_{\eta'} = 958$ MeV for $\overset{\circ}{m}_0$. Under the above assumptions, this is possible only if $650 \text{ MeV} < \overset{\circ}{m}_0 < 1.1 \text{ GeV}$. These bounds agree with those found in [22]. Using the value for the topological susceptibility given within this work which corresponds to $v_0^{(2)} = 0.003174 \text{ GeV}^4$ in our framework, we obtain $\overset{\circ}{m}_0 = 905$ MeV ($\overset{\circ}{m}_0^2 = 857$ MeV) for $f = 88$ MeV ($f = 93$ MeV).

The masses for the octet from eq. (21) are exactly the same as in $SU(3)$ perturbation theory provided the LECs are related by $\beta_k^{(0)} = \beta_k^{SU(3)}$ for $k = 4, 5, 6, 8$ and

$$\beta_7^{SU(3)} = \beta_7^{(0)} - \frac{(\tilde{v}_2^{(1)})^2}{3f^2 \overset{\circ}{m}_0^2}. \quad (25)$$

This is in agreement with the results from [23,24]. In [23] the η' field was integrated out explicitly to match the LECs to their $SU(3)$ values. Note that within the approach of large N_c chiral perturbation theory, $v_3^{(1)}$ is of higher order and does not appear at the order considered there. The phenomenological value of $\beta_7^{SU(3)} = (-0.35 \pm 0.2) \times 10^{-3}$ may be saturated completely by the additional term in eq. (25).

4.2 Decay constants

Phenomenologically the η - η' mixing can be extracted from the pseudoscalar decays. The decay constants F_{kl} are defined by the processes $\langle 0 | A_\mu^l | \phi_k \rangle = i p_\mu F_{kl}$. At lowest order the decay constants are $F_{kl} = f \delta_{kl}$ for the octet and

$$F_{\eta'0} = f_0 = \frac{\sqrt{6}\lambda(f^2 + 6v_5^{(0)})}{f^2} \quad (26)$$

for the singlet. At next-to-leading order there are also off-diagonal decay constants where mixing effects appear. The mixing will be parametrized by

$$\begin{aligned} (F_{\eta 8}, F_{\eta' 8}) &= F_8 (\cos \vartheta_8, \sin \vartheta_8), \\ (F_{\eta 0}, F_{\eta' 0}) &= F_0 (-\sin \vartheta_0, \cos \vartheta_0), \end{aligned} \quad (27)$$

while no other mixing occurs among the decay constants in the isospin limit $m_u = m_d$.

In this section we need a few more operators from the fourth-order Lagrangian (14), namely O_k with $k = 46, 47, 52, 53$. We find the decay constants at next-to-leading order in a similar way as the masses in sect. 3:

$$\begin{aligned} F_\pi &= f \left[1 + 12\beta_4^{(0)} \frac{\overset{\circ}{m}_8^2}{f^2} + 4\beta_5^{(0)} \frac{\overset{\circ}{m}_\pi^2}{f^2} - \frac{\Delta_\pi + \frac{1}{2}\Delta_K}{f^2} \right], \\ F_K &= f \left[1 + 12\beta_4^{(0)} \frac{\overset{\circ}{m}_8^2}{f^2} + 4\beta_5^{(0)} \frac{\overset{\circ}{m}_K^2}{f^2} - \frac{\frac{3}{8}\Delta_\pi + \frac{3}{4}\Delta_K + \frac{3}{8}\Delta_\eta}{f^2} \right], \\ F_{\eta 8} &= f \left[1 + 12\beta_4^{(0)} \frac{\overset{\circ}{m}_8^2}{f^2} + 4\beta_5^{(0)} \frac{\overset{\circ}{m}_\eta^2}{f^2} - \frac{\frac{3}{2}\Delta_K}{f^2} \right], \\ F_{\eta' 8} &= -\frac{2\sqrt{2}\overset{\circ}{m}_\Delta^2 \tilde{v}_2^{(1)}}{\overset{\circ}{m}_0^2 f}, \\ F_{\eta 0} &= \frac{2\sqrt{2}\overset{\circ}{m}_\Delta^2}{f^2} \left[\frac{f_0 \tilde{v}_2^{(1)}}{\overset{\circ}{m}_0^2} - \sqrt{6}\lambda(2\beta_{5,18} + 3\beta_{46}^{(0)} + 3\beta_{53}^{(0)}) \right], \\ F_{\eta' 0} &= f_0 + \frac{4\overset{\circ}{m}_8^2(2\sqrt{6}\lambda - f_0)\beta_{4,5,17,18}}{f^2} \\ &\quad + \frac{\sqrt{6}\lambda \overset{\circ}{m}_8^2}{f^2} (12\beta_{46}^{(0)} + 36\beta_{47}^{(0)} - 12\beta_{53}^{(0)} - 6\sqrt{6}\beta_{52}^{(1)}). \end{aligned} \quad (28)$$

In second order the two decay amplitudes F_8, F_0 are given by $F_8 = F_{\eta_8}$ and $F_0 = F_{\eta'0}$, while the angles ϑ_8, ϑ_0 are

$$\begin{aligned}\vartheta_8 &= -\frac{2\sqrt{2}\overset{\circ}{m}_\Delta^2}{\overset{\circ}{m}_0^2 f^2} \tilde{v}_2^{(1)}, \\ \vartheta_0 &= \vartheta_8 + \frac{2\sqrt{2}\overset{\circ}{m}_\Delta^2}{f^2 + 6v_5^{(0)}} (2\beta_{5,18} + 3\beta_{46}^{(0)} + 3\beta_{53}^{(0)}).\end{aligned}\quad (29)$$

This is the leading-order contribution to the mixing angles and both angles differ. Phenomenological values for the angles have been given, *e.g.* in [12]: $\vartheta_8 = -21.2^\circ$, $\vartheta_0 = -9.2^\circ$.

4.3 Fit

We will use the above equations to fit some of the parameters. To be more precise, we use the mass formula of the η' at second chiral order, the mixing angle ϑ_8 and assume the complete saturation of $\beta_7^{SU(3)}$ due to η - η' mixing (*i.e.* $\beta_7^{(0)} \approx 0$) in order to obtain values for the parameters $\overset{\circ}{m}_0^2$, $\tilde{v}_2^{(1)}$ and $\beta_{4,5,17,18}$. The $1/N_c$ estimate for $\tilde{v}_2^{(2)}$ reads $\tilde{v}_2^{(2)} \approx 2\tilde{v}_2^{(1)} - \frac{1}{4}f^2$. Taking the values $\overset{\circ}{m}_{\eta'} = 958$ MeV and $\vartheta_8 = -21.2^\circ$, the resulting parameters are

$$\begin{aligned}\overset{\circ}{m}_0 &= 847 \text{ MeV}, & \tilde{v}_2^{(1)} &= 1.25 \times \frac{1}{4}f^2, \\ \beta_{4,5,17,18} &= 0.47 \times 10^{-3}.\end{aligned}\quad (30)$$

These values are all in the expected ranges, however, they depend heavily on our assumptions: A change in $\overset{\circ}{m}_{\eta'}$ by 10 MeV, *e.g.*, requires $\beta_{4,5,17,18}$ to change by -0.15×10^{-3} , whereas a change in ϑ_8 by 1° or in β_7 by 0.05×10^{-3} results in the changes

$$\begin{aligned}\overset{\circ}{m}_0 &: + 6 \text{ MeV}, & \tilde{v}_2^{(1)} &: +0.10 \times \frac{1}{4}f^2, \\ \beta_{4,5,17,18} &: + 0.33 \times 10^{-3}, \\ \overset{\circ}{m}_0 &: + 59 \text{ MeV}, & \tilde{v}_2^{(1)} &: +0.18 \times \frac{1}{4}f^2, \\ \beta_{4,5,17,18} &: + 1.08 \times 10^{-3}.\end{aligned}\quad (31)$$

The value for $\overset{\circ}{m}_0$ is in agreement with the result given in [22].

At fourth chiral order the evaluation of the η' mass is rendered more difficult due to the proliferation of new counterterms. We will therefore make the following rough estimate by neglecting the unknown OZI-violating couplings and keeping only the known parameters. The terms of fourth chiral order for $m_{\eta'}$ in eq. (21) are then—in order of appearance—corrections of about -1% , 2.5% , 40% , 2% , 7% relatively to $\overset{\circ}{m}_{\eta'}$. The loop term delivers by far the greatest contribution but is highly scale dependent. This can be seen immediately, *e.g.*, by noting that the prefactor $3\overset{\circ}{m}_\pi^2 \Delta_\pi + 4\overset{\circ}{m}_K^2 \Delta_K + \overset{\circ}{m}_\eta^2 \Delta_\eta$ vanishes at a scale of about $\mu = 520$ MeV. The counterterms included in γ_1 and γ_2 which cancel this scale dependence will therefore also vary

strongly with μ and might lead to sizeable contributions depending on the choice for μ . In order to confine their approximate size, one must consider further processes involving these couplings.

5 Conclusions

In this investigation we have presented η - η' mixing up to one-loop order in the context of the masses and decay constants of the η - η' system. We worked in the framework of infrared regularized $U(3)$ chiral perturbation theory which permits a strict chiral counting scheme without employing large N_c counting rules. We treat the η' as a massive state, whereas it is considered to be a small quantity in large N_c chiral perturbation theory. It turns out that even at leading order the η and η' fields do not follow the usually assumed one-mixing-angle scheme. Already at tree level the mixing of these states cannot be parametrized by just one single angle which is in contradistinction to large N_c chiral perturbation theory where one mixing angle is sufficient at lowest order. In this framework the physical fields η and η' are related to the pure octet and singlet states, η_8 and η_0 , via a matrix which includes at leading order the parameter combinations $\tilde{v}_2^{(1)}$, $\beta_{5,18}$ and $\beta_{4,5,17,18}$ as well as $\overset{\circ}{m}_0$, the η' mass in the chiral limit of vanishing quark masses, see eq. (12). As an immediate consequence, matrix elements involving η and η' fields will include these parameter combinations and can be used to extract their numerical values by comparison with experimental data. The pseudoscalar decays, *e.g.*, are suited to obtain reasonable estimates for the couplings and a fit to the two angles ϑ_8 and ϑ_0 can be easily accommodated as shown in the preceding section. However, using the results from eq. (29), we can turn the argument around and obtain a rough estimate for ϑ_8 and ϑ_0 . To this end, we assume that the values of the $1/N_c$ suppressed couplings are negligible, *i.e.* in particular $|v_5^{(0)}| \ll f^2/4$, and $|\beta_{18}^{(0)}|, |\beta_{46}^{(0)}|, |\beta_{53}^{(0)}| \ll |\beta_5^{(0)}|$, hence generalizing our approximation for the OZI-violating contributions made in the last section. The parameter $\beta_5^{(0)}$ itself is phenomenologically determined by the ratio [24, 21]

$$\begin{aligned}\frac{F_K}{F_\pi} &= 1 + 4(m_K^2 - m_\pi^2) \frac{\beta_5^{(0)}}{f^2} + \frac{5}{8f^2} \Delta_\pi \\ &\quad - \frac{1}{4f^2} \Delta_K - \frac{3}{8f^2} \Delta_\eta \approx 1.22.\end{aligned}\quad (32)$$

Using $f = 88$ MeV yields the value $\beta_5^{(0)} = 1.3 \times 10^{-3}$ which is consistent with [21]. With these rough assumptions, we obtain

$$\vartheta_0 - \vartheta_8 = \frac{16\sqrt{2}(m_K^2 - m_\pi^2)}{3f^2} \beta_5^{(0)} = 16.4^\circ, \quad (33)$$

which slightly overestimates, *e.g.*, the phenomenological extraction of [12]. This indicates that other contributions such as the neglected LECs from eq. (29) or higher orders

may modify our estimate for $\vartheta_0 - \vartheta_8$; nevertheless, the two angles differ considerably. The result is similar to the one obtained in [9]. Note, however, that within the present scheme, this is the leading contribution, while in the combined chiral and $1/N_c$ expansion the difference of the two angles starts at subleading order and the form as given in eq. (33) corresponds even to next-to-next-to-leading order.

The phenomenological determination of η - η' mixing from photonic decays of the η and η' should yield a more reliable value. The lowest-order contribution to these decays originates from the anomalous Wess-Zumino-Witten term which is of fourth chiral order. In order to pin down the values of the two angles accurately, one must calculate $SU(3)$ breaking corrections to the Wess-Zumino-Witten term which are of sixth chiral order and beyond the scope of the present investigation.

Under reasonable assumptions for the parameters of the η' mass at second chiral order, we were able to obtain a range for \dot{m}_0 : $650 \text{ MeV} < \dot{m}_0 < 1.1 \text{ GeV}$, *i.e.* in our approach it is in principle possible that the η' mass contribution due to the axial $U(1)$ anomaly can be larger than the physical mass of 958 MeV and is lowered by leading-order symmetry-breaking terms. Comparing the mixing angle ϑ_8 with phenomenological analyses, and assuming that $\beta_7^{SU(3)}$ is completely saturated by the η' resonance, we were able to disentangle two of the parameters: $\tilde{v}_2^{(1)}$, which is predominantly responsible for η - η' mixing and $\dot{m}_0 \approx 850 \text{ MeV}$. This value for \dot{m}_0 is in agreement with other analyses (see, *e.g.*, [22]) and it shows that the saturation of $\beta_7^{SU(3)}$ was a consistent assumption.

The mass \dot{m}_0^2 is given by $\dot{m}_0^2 = 2v_0^{(2)}/f^2$, see eq. (10), a well-known result [25]. In the large N_c limit $v_0^{(2)}$ coincides with $3\tau_{\text{GD}}$, where τ_{GD} is the topological susceptibility of Gluodynamics. It represents the mean-square winding number per unit volume of Euclidean space

$$\tau_{\text{GD}} \equiv \int d^4x \langle 0 | T \omega(x) \omega(0) | 0 \rangle_{\text{GD}}, \quad (34)$$

with

$$\omega = \frac{g^2}{16\pi^2} \text{tr} G_{\mu\nu} \tilde{G}^{\mu\nu}. \quad (35)$$

The uncertainty in \dot{m}_0 translates immediately into a range for τ_{GD}

$$0.55 \times 10^{-3} \text{ GeV}^4 < \tau_{\text{GD}} < 1.56 \times 10^{-3} \text{ GeV}^4. \quad (36)$$

However, some of the results are rather sensitive to the assumptions made for the parameters. A further study of the η - η' system, such as their hadronic decay modes as well as the anomalous decays, should yield more reliable values for some parameters and the mixing angles as they appear in the parametrization of the pseudoscalar decay constants for the η and η' [19].

We would like to thank Stefan Wetzel for reading the manuscript. This work is supported in part by the DFG.

Appendix A. Fourth-order operators

We use the standard definitions of chiral perturbation theory

$$\begin{aligned} F_{\mu\nu}^L &= \partial_\mu \tilde{l}_\nu - \partial_\nu \tilde{l}_\mu - i[\tilde{l}_\mu, \tilde{l}_\nu], \\ F_{\mu\nu}^R &= \partial_\mu \tilde{r}_\nu - \partial_\nu \tilde{r}_\mu - i[\tilde{r}_\mu, \tilde{r}_\nu], \\ \chi &= 2B(s + ip), \end{aligned} \quad (A.1)$$

where U is a unitary matrix containing the meson fields. The fields s , p , $v_\mu = \frac{1}{2}(\tilde{r}_\mu + \tilde{l}_\mu)$ and $\tilde{a}_\mu = \frac{1}{2}(\tilde{r}_\mu - \tilde{l}_\mu)$ are the external sources that couple to the QCD Lagrangian. The singlet axial-vector source $\langle \tilde{a}_\mu \rangle$ has been rescaled to account for the dependence on the running QCD scale, eq. (4).

We make use of the following abbreviations for the definition of the fourth-order Lagrangian:

$$\begin{aligned} C_\mu &= U^\dagger D_\mu U, \\ T_\mu &= iD_\mu \theta, \\ M &= U^\dagger \chi + \chi^\dagger U, \\ N &= U^\dagger \chi - \chi^\dagger U, \\ F_{\mu\nu}^\pm &= F_{\mu\nu}^L \pm U^\dagger F_{\mu\nu}^R U. \end{aligned} \quad (A.2)$$

The fourth-order operators O_k are given in table 1 and the fourth-order Lagrangian

$$\mathcal{L}^{(4)} = \sum_k \beta_k O_k \quad (A.3)$$

is a sum of these operators coupled with functions β_k of the invariant $\eta_0 + \sqrt{\lambda}\theta$, which can be expanded as in eq. (5).

In standard $SU(3)$ ($U(3)$) chiral perturbation theory the fourth-order Lagrangian consists of all possible fourth-order operators with coupling constants (functions in $\eta_0 + \sqrt{\lambda}\theta$) not fixed by chiral symmetry. In total there are 13 (58) independent fourth-order operators, one of which can be eliminated by the Cayley-Hamilton identity for $n_l = 3$ light flavors. The equation of motion for the meson fields has been used extensively in order to eliminate operators involving the divergence $D^\mu C_\mu$.

In the renormalization scheme presented in Appendix B, we will treat the η_0 as a background field that is not restricted by an equation of motion. To this end, we need to include additional operators involving $\langle D^\mu C_\mu \rangle$. To second order the only new counterterm is proportional to $\langle D^\mu C_\mu \rangle$ which is a total divergence and equivalent to operators already present in the second-order Lagrangian. It can therefore be omitted. At fourth order, however, new operators must be included which have not been considered in previous approaches. The eight additional counterterms read

$$\begin{aligned} O_{58} &= i \langle D^\mu C_\mu \rangle \langle C^\nu C_\nu \rangle, & O_{59} &= i \langle D^\mu C_\mu \rangle \langle C^\nu \rangle \langle C_\nu \rangle, \\ O_{60} &= i \langle D^\mu C_\mu \rangle \langle C^\nu \rangle T_\nu, & O_{61} &= i \langle D^\mu C_\mu \rangle T^\nu T_\nu, \\ O_{62} &= \langle D^\mu C_\mu \rangle \langle D^\nu C_\nu \rangle, & O_{63} &= \langle D^\mu C_\mu \rangle D^\nu T_\nu, \\ O_{64} &= i \langle D^\mu C_\mu \rangle \langle M \rangle, & O_{65} &= \langle D^\mu C_\mu \rangle \langle N \rangle. \end{aligned} \quad (A.4)$$

Table 1. Fourth-order operators in $U(3)$ ChPT [17].

$O_0 = \langle C^\mu C^\nu C_\mu C_\nu \rangle,$	$O_1 = \langle C^\mu C_\mu \rangle \langle C^\nu C_\nu \rangle,$
$O_2 = \langle C^\mu C^\nu \rangle \langle C_\mu C_\nu \rangle,$	$O_3 = \langle C^\mu C_\mu C^\nu C_\nu \rangle,$
$O_{13} = -\langle C^\mu \rangle \langle C_\mu C^\nu C_\nu \rangle,$	$O_{14} = -\langle C^\mu \rangle \langle C_\mu \rangle \langle C^\nu C_\nu \rangle,$
$O_{15} = -\langle C^\mu \rangle \langle C^\nu \rangle \langle C_\mu C_\nu \rangle,$	$O_{16} = \langle C^\mu \rangle \langle C_\mu \rangle \langle C^\nu \rangle \langle C_\nu \rangle,$
$O_4 = -\langle C^\mu C_\mu \rangle \langle M \rangle,$	$O_5 = -\langle C^\mu C_\mu M \rangle,$
$O_{17} = \langle C^\mu \rangle \langle C_\mu \rangle \langle M \rangle,$	$O_{18} = -\langle C^\mu \rangle \langle C_\mu M \rangle,$
$O_{21} = \langle C^\mu C_\mu iN \rangle,$	$O_{22} = \langle C^\mu C_\mu \rangle \langle iN \rangle,$
$O_{23} = \langle C^\mu \rangle \langle C_\mu iN \rangle,$	$O_{24} = \langle C^\mu \rangle \langle C_\mu \rangle \langle iN \rangle,$
$O_6 = \langle M \rangle \langle M \rangle,$	$O_7 = \langle N \rangle \langle N \rangle,$
$O_8 = \frac{1}{2} \langle MM + NN \rangle,$	$O_{12} = \frac{1}{4} \langle MM - NN \rangle,$
$O_{25} = \langle iMN \rangle,$	$O_{26} = \langle M \rangle \langle iN \rangle,$
$O_9 = i \langle C_\mu C_\nu F_+^{\mu\nu} \rangle,$	$O_{27} = \langle C_\mu \rangle \langle C_\nu F_-^{\mu\nu} \rangle,$
$O_{29} = i \varepsilon_{\mu\nu\rho\sigma} \langle C^\mu C^\nu F_+^{\rho\sigma} \rangle,$	$O_{30} = \varepsilon_{\mu\nu\rho\sigma} \langle C^\mu \rangle \langle C^\nu F_-^{\rho\sigma} \rangle,$
$O_{10} = \frac{1}{4} \langle F_+^{\mu\nu} F_{\mu\nu}^+ - F_-^{\mu\nu} F_{\mu\nu}^- \rangle,$	$O_{11} = \frac{1}{2} \langle F_+^{\mu\nu} F_{\mu\nu}^+ + F_-^{\mu\nu} F_{\mu\nu}^- \rangle,$
$O_{20} = \frac{1}{4} \langle F_+^{\mu\nu} \rangle \langle F_{\mu\nu}^+ \rangle - \frac{1}{4} \langle F_-^{\mu\nu} \rangle \langle F_{\mu\nu}^- \rangle,$	$O_{19} = \frac{1}{2} \langle F_+^{\mu\nu} \rangle \langle F_{\mu\nu}^+ \rangle + \frac{1}{2} \langle F_-^{\mu\nu} \rangle \langle F_{\mu\nu}^- \rangle,$
$O_{28} = \frac{1}{4} \varepsilon_{\mu\nu\rho\sigma} \langle F_+^{\mu\nu} F_+^{\rho\sigma} - F_-^{\mu\nu} F_-^{\rho\sigma} \rangle,$	
$O_{31} = T^\mu \langle C_\mu C^\nu C_\nu \rangle,$	$O_{32} = T^\mu \langle C_\mu \rangle \langle C^\nu C_\nu \rangle,$
$O_{33} = T^\mu \langle C_\mu C^\nu \rangle \langle C_\nu \rangle,$	$O_{34} = T^\mu \langle C_\mu \rangle \langle C^\nu \rangle \langle C_\nu \rangle,$
$O_{35} = T^\mu T_\mu \langle C^\nu C_\nu \rangle,$	$O_{37} = T^\mu T_\mu \langle C^\nu \rangle \langle C_\nu \rangle,$
$O_{36} = T^\mu T^\nu \langle C_\mu C_\nu \rangle,$	$O_{38} = T^\mu T^\nu \langle C_\mu \rangle \langle C_\nu \rangle,$
$O_{39} = T^\mu T_\mu T^\nu \langle C_\nu \rangle,$	$O_{40} = T^\mu T_\mu T^\nu T_\nu,$
$O_{41} = i D^\mu T_\mu \langle C^\nu C_\nu \rangle,$	$O_{42} = i D^\mu T_\mu \langle C^\nu \rangle \langle C_\nu \rangle,$
$O_{43} = i D^\mu T_\mu T^\nu \langle C_\nu \rangle,$	$O_{44} = i D^\mu T_\mu T^\nu T_\nu,$
$O_{45} = D^\mu T_\mu D^\nu T_\nu,$	
$O_{46} = T^\mu \langle C_\mu M \rangle,$	$O_{47} = T^\mu \langle C_\mu \rangle \langle M \rangle,$
$O_{48} = T^\mu \langle C_\mu iN \rangle,$	$O_{49} = T^\mu \langle C_\mu \rangle \langle iN \rangle,$
$O_{50} = T^\mu T_\mu \langle M \rangle,$	$O_{51} = T^\mu T_\mu \langle iN \rangle,$
$O_{52} = i D^\mu T_\mu \langle M \rangle,$	$O_{53} = D^\mu T_\mu \langle N \rangle,$
$O_{54} = T_\mu \langle C_\nu F_-^{\mu\nu} \rangle,$	$O_{55} = T_\mu \langle C_\nu \rangle \langle F_-^{\mu\nu} \rangle,$
$O_{56} = -\varepsilon_{\mu\nu\rho\sigma} T^\mu \langle C^\nu F_-^{\rho\sigma} \rangle,$	$O_{57} = -\varepsilon_{\mu\nu\rho\sigma} T^\mu \langle C^\nu \rangle \langle F_-^{\rho\sigma} \rangle.$

These operators are needed as long as the phase of U which describes the singlet field is treated as a background field. When subsequently the phase of U is dealt with as a propagating field, its equation of motion [17] may be used to eliminate the new operators. The amplitudes are then renormalizable only on-shell, but if one prefers to keep the new operators instead, they are renormalizable even if the η' field is off-shell. We have confirmed this property for a number of amplitudes.

Appendix B. Renormalization

In this section, we work out the renormalization of the one-loop functional of the Goldstone boson loops proceeding along the lines of [17] and using their notation. We will sketch the method briefly and highlight the differences since the details can be found in [17,24]. The alternative treatment of the singlet field within our approach yields substantially different results. In the scheme of [17] the singlet is a quantum field, whereas we treat it as an

external field which does not propagate so that we can restrict ourselves to $SU(3)$ instead of $U(3)$ matrices and relations. The results of this appendix can be used as a check for the infrared regularized loop contributions in the present investigation since the η' does not appear inside loops at the order we are working. The employed $SU(3)$ relations are

$$\lambda^a A \lambda^a = 2 \langle A \rangle - (2/n_l) A,$$

$$\langle \lambda^a A \rangle \langle \lambda^a B \rangle = 2 \langle AB \rangle - (2/n_l) \langle A \rangle \langle B \rangle.$$

In the scheme of [24], on the other hand, the singlet field is not included explicitly but the methodology to extract the divergences is equivalent. Omitting in the present investigation the external singlet field contributions reproduces the results of [24].

We start by introducing a background field $\bar{U} \in U(3)$ which obeys the equation of motion for the octet whereas its phase is arbitrary. The matrix U is decomposed as $U = \bar{U} \exp(i\Delta)$ with quantum fluctuations $\Delta \in SU(3)$. The second-order chiral Lagrangian expanded up to two

powers of Δ reads

$$\mathcal{L}(U) = \mathcal{L}(\bar{U}) + V_1(X)\langle D^\mu \Delta D_\mu \Delta \rangle + V_1(X)\langle C^\mu[\Delta, D_\mu \Delta] \rangle - \frac{1}{2}V_2(X)\langle \Delta^2 M \rangle - \frac{1}{2}iV_3(X)\langle \Delta^2 N \rangle, \quad (\text{B.1})$$

with $D_\mu \Delta = \partial_\mu \Delta - i[l_\mu, \Delta]$ and the invariant quantity $X = \langle \log \bar{U} \rangle + i(\sqrt{6}\lambda/f)\theta = i\sqrt{6}(\eta_0 + \sqrt{\lambda}\theta)/f$. The terms linear in Δ vanish upon using the equation of motion and we drop the piece $\mathcal{L}(\bar{U})$ which does not depend on the quantum fluctuations Δ . Equation (B.1) corresponds to eq. (21) in [17] when all terms proportional to $\langle \Delta \rangle$ are neglected since they vanish for $\Delta \in SU(3)$. We then set $\Delta = \varphi^a \lambda^a / 2\sqrt{V_1}$ to obtain canonically normalized kinetic terms for the octet φ . After partial integration and completion of a square, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}d^\mu \varphi^a d_\mu \varphi^a - \frac{1}{2}\varphi^a \sigma^{ab} \varphi^b. \quad (\text{B.2})$$

The connection ω of $d_\mu \varphi^a = \partial_\mu \varphi^a + \omega_\mu^{ab} \varphi^b$, the curvature R thereof and the mass term σ read

$$\begin{aligned} \omega_\mu^{ab} &= \frac{1}{2}i\langle \omega_\mu[\lambda^a, \lambda^b] \rangle, \\ R_{\mu\nu}^{ab} &= \frac{1}{2}i\langle R_{\mu\nu}[\lambda^a, \lambda^b] \rangle, \\ \omega_\mu &= l_\mu + \frac{1}{2}iC_\mu, \\ R_{\mu\nu} &= \frac{1}{2}F_{\mu\nu}^L + \frac{1}{2}U^\dagger F_{\mu\nu}^R U - \frac{1}{4}i[C_\mu, C_\nu], \\ \sigma^{ab} &= \frac{1}{8}\langle [C^\mu, \lambda^a][C_\mu, \lambda^b] \rangle \\ &\quad + \frac{1}{8}\langle (\omega_2 M + i\omega_3 N)\{\lambda^a, \lambda^b\} \rangle + \delta^{ab}S, \\ S &= -\frac{1}{2}\omega_1'' \partial^\mu X \partial_\mu X + \frac{1}{4}\omega_1' \omega_1' \partial^\mu X \partial_\mu X \\ &\quad - \frac{1}{2}\omega_1' \partial^\mu \partial_\mu X, \end{aligned} \quad (\text{B.3})$$

where we have suppressed the bars in \bar{U} and in the related quantities of eq. (A.2). The functions ω_k , ω_1' and ω_1'' are defined as the quotients V_k/V_1 , V_1'/V_1 and V_1''/V_1 , respectively. Note that the derivate V_i' is defined as in [17] as

$$V_1' = \frac{\partial V_1}{\partial X} = \frac{1}{i\sqrt{6}} \frac{\partial V_1}{\partial(\eta_0/f)}, \quad (\text{B.4})$$

in comparison to eq. (5). The differences to eqs. (27-30) in [17] stem from the modified algebra.

Taking the fields φ as quantum fields, whereas \bar{U} , X are external fields, we calculate the one-loop effective action. The divergent piece in dimensional regularization at $d = 4$ is

$$\frac{\Gamma}{(4\pi)^2(4-d)}, \quad \Gamma = \frac{1}{12}R_{ab}^{\mu\nu}R_{\mu\nu}^{ba} + \frac{1}{2}\sigma_{ab}\sigma_{ba}. \quad (\text{B.5})$$

After some algebra we obtain Γ expressed in terms of the 58 known and 8 new fourth-order operators (cf. Ap-

pendix A)

$$\begin{aligned} \Gamma &= \frac{1}{48}(n_l O_0 + 3O_1 + 6O_2 + 2n_l O_3 + 12O_{13}) \\ &\quad + \frac{1}{24}(2n_l O_9 - 2n_l O_{10} - n_l O_{11} + O_{19} + 2O_{20}) \\ &\quad + \frac{1}{8}(-\omega_2 O_4 + n_l \omega_2 O_5 - 2\omega_2 O_{18} - n_l \omega_3 O_{21} \\ &\quad \quad - \omega_3 O_{22} + 2\omega_3 O_{23}) \\ &\quad + \frac{n_l^2 + 2}{16n_l^2}(\omega_2 \omega_2 O_6 + \omega_3 \omega_3 O_7 + 2\omega_2 \omega_3 O_{26}) \\ &\quad + \frac{n_l^2 - 4}{16n_l}((\omega_2 \omega_2 - \omega_3 \omega_3)O_8 + (\omega_2 \omega_2 + \omega_3 \omega_3)O_{12} \\ &\quad \quad + 2\omega_2 \omega_3 O_{25}) \\ &\quad + \frac{\omega_1' \omega_1' - 2\omega_1''}{8}(n_l O_{14} + O_{16}^* - 2n_l O_{32} + 2O_{34}^* \\ &\quad \quad - n_l O_{35} + O_{37}^*) \\ &\quad + \frac{(n_l^2 - 1)(\omega_1' \omega_1' - 2\omega_1'')}{8n_l}(\omega_2(O_{17} + 2O_{47} + O_{50}) \\ &\quad \quad + \omega_3(O_{24} + 2O_{49} + O_{51})) \\ &\quad + \frac{i\omega_1'}{4}(-n_l O_{41} + O_{42}^* - n_l O_{58} + O_{59}^*) \\ &\quad + \frac{(n_l^2 - 1)i\omega_1'}{4n_l}(\omega_2 O_{52} - \omega_3 O_{53} + \omega_2 O_{64} - \omega_3 O_{65}) \\ &\quad + \frac{(n_l^2 - 1)(\omega_1' \omega_1' - 2\omega_1'')^2}{32}(O_{16}^* + 4O_{34}^* + 2O_{37}^* \\ &\quad \quad + 4O_{38} + 4O_{39} + O_{40}) \\ &\quad + \frac{(n_l^2 - 1)i\omega_1'(\omega_1' \omega_1' - 2\omega_1'')}{8}(-O_{42}^* - 2O_{43} \\ &\quad \quad - O_{44} - O_{59}^* - 2O_{60} - O_{61}) \\ &\quad + \frac{(n_l^2 - 1)\omega_1' \omega_1'}{8}(O_{45} + O_{62} + 2O_{63}). \end{aligned} \quad (\text{B.6})$$

The divergence of the one-loop effective action needs to be cancelled by counterterms in the coupling functions of the fourth-order operators. The corresponding renormalization functions can be read off from the coefficients of the operators in Γ . For convenience the operators which appear twice are marked by *.

The structure of Γ equals that of standard $SU(3)$ chiral perturbation theory if $\omega_2 = 1$, $\omega_3 = \omega_1' = \omega_1'' = 0$ and all non-standard operators ignored. For $n_l = 3$ the Cayley-Hamilton matrix identity can be used to shuffle the coefficient of O_0 to those of O_k with $k = 1, 2, 3$ (and 13, 14, 15, 16). The result for Γ has been confirmed by calculating four point amplitudes such as $\eta'\eta' \rightarrow \eta'\eta'$ scattering. After performing the renormalization prescription as given by eq. (B.6), the amplitudes were rendered finite and independent of the scale μ introduced in dimensional regularization.

References

1. R. Akhoury, J.M. Frère, Phys. Lett. B **220**, 258 (1989).
2. E.P. Venugopal, B.R. Holstein, Phys. Rev. D **57**, 4397 (1998).
3. F.J. Gilman, R. Kauffman, Phys. Rev. D **36**, 2761 (1987).
4. A. Bramon, R. Escribano, M.D. Scadron, Phys. Lett. B **403**, 339 (1997).
5. P. Ball, J.M. Frère, M. Tytgat, Phys. Lett. B **365**, 367 (1996).
6. A. Bramon, R. Escribano, M.D. Scadron, Eur. Phys. J. C **7**, 271 (1999).
7. J.F. Donoghue, E. Golowich, B.R. Holstein, *Dynamics of the Standard Model* (Cambridge University Press, Cambridge, 1992).
8. H. Leutwyler, Nucl. Phys. Proc. Suppl. **64**, 223 (1998).
9. R. Kaiser, H. Leutwyler, *Pseudoscalar decay constants at large $N(c)$* , in *Proceedings of the Workshop Nonperturbative Methods in Quantum Field Theory*, edited by A.W. Schreiber, A.G. Williams, A.W. Thomas (World Scientific, Singapore, 1998).
10. R. Kaiser, H. Leutwyler, Eur. Phys. J. C **17**, 623 (2000).
11. T. Feldmann, P. Kroll, Eur. Phys. J. C **5**, 327 (1998).
12. T. Feldmann, P. Kroll, B. Stech, Phys. Rev. D **58**, 114006 (1998).
13. R. Escribano, J.M. Frère, Phys. Lett. B **459**, 288 (1999).
14. P. Herrera-Siklody, J.I. Latorre, P. Pascual, J. Taron, Phys. Lett. B **419**, 326 (1998).
15. B. Borasoy, S. Wetzel, Phys. Rev. D **63**, 074019 (2001).
16. T. Becher, H. Leutwyler, Eur. Phys. J. C **9**, 643 (1999).
17. P. Herrera-Siklody, J.I. Latorre, P. Pascual, J. Taron, Nucl. Phys. B **497**, 345 (1997).
18. P. Herrera-Siklody, hep-ph/9902446.
19. N. Beisert, B. Borasoy, *The $\eta' \rightarrow \eta\pi\pi$ decay in $U(3)$ chiral perturbation theory*, submitted to Nucl. Phys. A, preprint TU München: TUM-T39-01-18.
20. P. Gerber, H. Leutwyler, Nucl. Phys. B **321**, 387 (1989).
21. J. Bijnens, G. Ecker, J. Gasser, *Chiral Perturbation Theory in 2nd DAPHNE Physics Handbook*, edited by L. Maiani, G. Pancheri, N. Paver (INFN, Frascati, 1995).
22. V. Gogohia, hep-ph/0012128.
23. P. Herrera-Siklody, Phys. Lett. B **442**, 359 (1998).
24. J. Gasser, H. Leutwyler, Nucl. Phys. B **250**, 465 (1985).
25. E. Witten, Ann. Phys. (N.Y.) **128**, 363 (1980); P. Di Vecchia, G. Veneziano, Nucl. Phys. B **171**, 253 (1980).